

# TORIC HIRZEBRUCH-RIEMANN-ROCH VIA ISHIDA'S THEOREM ON THE TODD GENUS

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ABSTRACT. We give a simple proof of the Hirzebruch-Riemann-Roch theorem for smooth complete toric varieties, based on Ishida's result in [5] that the Todd genus of a smooth complete toric variety is one.

## 1. INTRODUCTION

The Hirzebruch-Riemann-Roch theorem relates the Euler characteristic of a coherent sheaf  $\mathcal{F}$  on a smooth complete  $n$ -dimensional variety  $X$  to intersection theory, via the formula

$$(1) \quad \chi(\mathcal{F}) = \int ch(\mathcal{F})Td(\mathcal{T}_X).$$

In [2], Brion-Vergne prove an equivariant Hirzebruch-Riemann-Roch theorem for complete simplicial toric varieties. If the toric variety is actually smooth, it is possible to derive (1) from their result. In this note, we give a simple direct proof of (1) when  $X$  is a smooth complete toric variety. Such a variety is determined by a smooth complete rational polyhedral fan  $\Sigma \subseteq N_{\mathbb{R}}$ , where  $N \simeq \mathbb{Z}^n$  is a lattice; we write  $X$  for the associated toric variety  $X_{\Sigma}$ . We will make use of the following standard facts about toric varieties. First,

$$(2) \quad Td(X_{\Sigma}) = \prod_{\rho \in \Sigma(1)} \frac{D_{\rho}}{1 - e^{-D_{\rho}}},$$

where  $\Sigma(k)$  denotes the set of  $k$ -dimensional faces of  $\Sigma$ . For  $\tau \in \Sigma(k)$  there is an associated torus invariant orbit  $O(\tau)$ , and we use  $V(\tau)$  to denote the orbit closure  $\overline{O(\tau)}$ , which has dimension  $n - k$ . A key fact is that (see [4], Proposition 3.2.7)

$$(3) \quad V(\tau) = \overline{O(\tau)} \simeq X_{\text{Star}(\tau)}.$$

Since  $\Sigma$  is smooth, all orbits are also smooth, and if  $\rho_i, \rho_j$  are distinct elements of  $\Sigma(1)$ , then (see [4], Lemma 12.5.7)

$$[D_{\rho_i}|_{V(\rho_j)}] = \begin{cases} V(\tau) & \tau = \rho_i + \rho_j \in \Sigma \\ 0 & \rho_i, \rho_j \text{ are not both in any cone in } \Sigma. \end{cases}$$

The final ingredient we need is a result of Ishida: building on work of Brion [1], in [5] Ishida shows that (1) holds for the structure sheaf of a smooth complete toric variety  $X$ :

$$(4) \quad 1 = \int Td(\mathcal{T}_X) = \left[ \prod_{\rho \in \Sigma(1)} \frac{D_{\rho}}{1 - e^{-D_{\rho}}} \right]_n.$$

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## 2. THE PROOF

For a smooth complete toric variety, any coherent sheaf has a resolution by line bundles [3], so it suffices to consider the case  $\mathcal{F} = \mathcal{O}_X(D)$ . Let  $X = X_\Sigma$ , and recall that  $\text{Pic}(X)$  is generated by the classes of the divisors  $D_\rho$ ,  $\rho \in \Sigma(1)$ . We will show that if (1) holds for a divisor  $D$ , then it also holds for  $D + D_\rho$  and  $D - D_\rho$ , for any  $\rho \in \Sigma(1)$ . We begin with the case  $D - D_\rho$ , and induct on the dimension of  $X$ .

A smooth complete toric variety of dimension one is simply  $\mathbb{P}^1$ , so the base case holds by Riemann-Roch for curves. Suppose the theorem holds for all smooth complete fans of dimension  $< n$ , and let  $\Sigma$  be a smooth complete fan of dimension  $n$ . When  $D = 0$  the result holds by Ishida's theorem. Let  $\rho \in \Sigma(1)$ , and partition the rays of  $\Sigma$  as

$$\Sigma(1) = \rho \cup \Sigma'(1) \cup \Sigma''(1),$$

where the rays in  $\Sigma'(1)$  are in one to one correspondence with the rays of the fan  $\text{Star}(\rho)$ . Let  $X' = X_{\text{Star}(\rho)} \simeq V(\rho)$ . Tensoring the standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D_\rho) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X'} \longrightarrow 0$$

with  $\mathcal{O}_X(D)$  yields the sequence

$$0 \longrightarrow \mathcal{O}_X(D - D_\rho) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_{X'}(D) \longrightarrow 0.$$

From the additivity of the Euler characteristic, we have

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(D - D_\rho)) = \chi(\mathcal{O}_{X'}(D)).$$

Our hypotheses imply that

$$\begin{aligned} \int_{X'} e^D Td(\mathcal{T}_{X'}) &= \chi(\mathcal{O}_{X'}(D)) \\ \int_X e^D Td(\mathcal{T}_X) &= \chi(\mathcal{O}_X(D)), \end{aligned}$$

so it suffices to show that

$$\begin{aligned} \int_{X'} ch(D) Td(\mathcal{T}_{X'}) &= \int_X (e^D - e^{D-D_\rho}) Td(\mathcal{T}_X) \\ (5) \qquad \qquad \qquad &= \int_X e^D \left( \frac{1 - e^{-D_\rho}}{D_\rho} \right) D_\rho Td(\mathcal{T}_X) \end{aligned}$$

Break the Todd class of  $X$  into two parts:

$$Td(\mathcal{T}_X) = \prod_{\gamma \in \Sigma'(1) \cup \rho} \frac{D_\gamma}{1 - e^{-D_\gamma}} \cdot \prod_{\gamma \in \Sigma''(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}}$$

In (5), the term  $\frac{1 - e^{-D_\rho}}{D_\rho}$  cancels with the corresponding term in  $Td(\mathcal{T}_X)$ , so that

$$\begin{aligned} \int_X e^D \left( \frac{1 - e^{-D_\rho}}{D_\rho} \right) D_\rho Td(\mathcal{T}_X) &= \int_X e^D D_\rho \prod_{\gamma \in \Sigma'(1) \cup \Sigma''(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}} \\ (6) \qquad \qquad \qquad &= \int_X e^D D_\rho \prod_{\gamma \in \Sigma'(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}}. \end{aligned}$$

The second equality follows since  $D_\rho \cdot D_\gamma = 0$  if  $\gamma \in \Sigma''(1)$ . By smoothness, all intersections are either zero or one, and thus

$$\begin{aligned} \int_X e^{D_\rho} \prod_{\gamma \in \Sigma'(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}} &= \left[ e^{D_\rho} \prod_{\gamma \in \Sigma'(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}} \right]_n \\ &= \left[ e^{D|_{V(\rho)}} \prod_{\gamma \in \Sigma'(1)} \frac{D_\gamma}{1 - e^{-D_\gamma}} \right]_{n-1} \\ &= \int_{X'} e^D \cdot Td(\mathcal{T}_{X'}). \end{aligned}$$

This proves the result for  $D - D_\rho$ . For  $D + D_\rho$ , the result follows using the substitution  $e^{D_\rho} - 1 = e^{D_\rho}(1 - e^{-D_\rho})$ .

**Question** Ishida's proof (4) is not easy; does there exist a simple proof of (4)?

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